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Inverse Beta and Generalized Bleimann–Butzer–Hahn Operators*

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In this paper we introduce two new Bernstein-type operators which are closely related to each other. The former is associated with the Pólya distribution and includes as a particular case the Bleimann–Butzer–Hahn operator. The second is associated with the inverse beta probability distribution. Approximation properties for both operators concerning rates of convergence, preservation of Lipschitz constants, and monotonic convergence under convexity are given. In dealing with the last two topics, probabilistic methods play an important role.

1. INTRODUCTION

A new Bernstein-type operator acting on real functions on the semi-axis $[0, \infty)$ is defined by

$$L_n^\alpha(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) w_{n,k}(x;\alpha), \qquad x \geqslant 0, n = 1, 2, ...,$$

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where α is a nonnegative parameter and

$$w_{n,k}(x;\alpha) := \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+i\alpha)\prod_{j=0}^{n-k-1}(1+j\alpha)}{\prod_{r=0}^{n-1}(1+x+r\alpha)}.$$

Note that we can write

$$L_n^\alpha(f,x) = Ef\left(\frac{U_n^{x,\alpha}}{n-U_n^{x,\alpha}+1}\right),$$

where E denotes mathematical expectation and $U_n^{x,\alpha}$ is a random variable having the Pólya–Eggenberger distribution with parameters n, x, 1, α [10]. If $\alpha = 0$, $U_n^{x,0}$ has the binomial distribution with parameters n, $x/(1+x)$ and L_n^0 is actually the operator introduced by Bleimann, Butzer, and Hahn [2], the approximation properties of which have been extensively studied in the literature [2, 5, 6, 14, 15, 20]. In what follows it is simply denoted by L_n.

Moreover, for any $x>0$, $\alpha>0$, and $n=1, 2, \ldots$, we have

$$w_{n,k}(x;\alpha) = \binom{n}{k}\int_0^\infty \frac{\theta^k}{(1+\theta)^n} h_\alpha^x(\theta)\,d\theta,$$

where h_α^x denotes the probability density of the inverse beta distribution (sometimes called *beta – prime* distribution) with parameters x/α, $1/\alpha$ [8, 11], that is,

$$h_\alpha^x(\theta) := \left(B\left(\frac{x}{\alpha}, \frac{1}{\alpha}\right)\right)^{-1} \frac{\theta^{x/\alpha-1}}{(1+\theta)^{(1+x)/\alpha}}, \qquad \theta>0,$$

where $B(\cdot,\cdot)$ is the beta function. Therefore, the representation

$$L_n^\alpha(f,x) = \int_0^\infty L_n(f,\theta)\,h_\alpha^x(\theta)\,d\theta \tag{1}$$

holds for any real function f on $[0, \infty)$.

This leads us to consider the integral operator T^α defined by

$$T^\alpha(f,x) := Ef(Z_\alpha^x), \tag{2}$$

where Z_α^x is a nonnegative random variable having the density h_α^x above. The expression (2) is well defined provided that f is a real measurable function on $(0, \infty)$ such that $E|f(Z_\alpha^x)| < \infty$. If f is defined on $[0, \infty)$ we write $T^\alpha(f, 0) := f(0)$.

Observe that

$$EZ_\alpha^x = \frac{x}{1-\alpha}, \qquad 0 < \alpha < 1, \tag{3}$$

and

$$D(x;\alpha) := E(Z_\alpha^x - x)^2 = \frac{\alpha x(1 + x + 2\alpha x)}{(1-\alpha)(1-2\alpha)}, \qquad 0 < \alpha < \frac{1}{2}. \tag{4}$$

Thus, from classical arguments, we have $T^\alpha(f, x) \to f(x)$ (as $\alpha \to 0$) whenever f is a real measurable bounded function on $(0, \infty)$ which is continuous at x. Estimates for $T^\alpha(f, x) - f(x)$ can be obtained by using standard methods. In particular, we have for any $x > 0$ and $0 < \alpha < \frac{1}{2}$

$$|T^\alpha(f, x) - f(x)| \leqslant 2\omega(f; (D(x;\alpha))^{1/2}), \tag{5}$$

for each $f \in C(0, \infty)$ such that $T^\alpha(|f|, x) < \infty$, and

$$|T^\alpha(f, x) - f(x)| \leqslant |f'(x)| \frac{\alpha x}{1-\alpha} + 2(D(x;\alpha))^{1/2}\, \omega(f'; (D(x;\alpha))^{1/2}), \tag{6}$$

if, in addition, $f' \in CB(0, \infty)$.

Coming back to the operator L_n^α, it is clear that

$$L_n^\alpha f = T^\alpha(L_n f), \tag{7}$$

for any real function f on $[0, \infty)$. This formula is analogous to that relating Bernstein, beta, and Stancu–Mühlbach operators [4, 17]. The following approximation properties are easily derived from (7): If f is a non-increasing convex function on $[0, \infty)$, then $L_n f \geqslant L_{n+1} f$ (cf. [15]) and therefore, for any $\alpha > 0$, we have $L_n^\alpha f \geqslant L_{n+1}^\alpha f$.

Moreover

$$L_n^\alpha(f, x) - T^\alpha(f, x) = \int_0^\infty (L_n(f, \theta) - f(\theta))\, h_\alpha^x(\theta)\, d\theta. \tag{8}$$

Thus, the bounds for $L_n(f, \theta) - f(\theta)$ given in [14, 15] can be used to obtain the following estimates:

For any $x > 0$, $0 < \alpha < \frac{1}{3}$ and $n = 1, 2, ...$ we have

$$|L_n^\alpha(f, x) - T^\alpha(f, x)| \leqslant (1 + (H(x;\alpha))^{1/2})\, \omega(f; n^{-1/2}), \tag{9}$$

for every $f \in C[0, \infty)$ such that $T^\alpha(|f|, x) < \infty$. If, in addition, $f' \in CB[0, \infty)$, we have

$$|L_n^\alpha(f, x) - T^\alpha(f, x)| \leqslant \frac{K(x;\alpha)}{n} \|f'\| + ((H(x;\alpha))^{1/2} + H(x;\alpha))\, n^{-1/2} \omega(f'; n^{-1/2}), \quad (10)$$

where

$$H(x;\alpha) := \frac{4x(1+x-\alpha)(1+x-2\alpha)}{(1-\alpha)(1-2\alpha)(1-3\alpha)}, \qquad K(x;\alpha) := \frac{x(1+x-\alpha)}{(1-\alpha)(1-2\alpha)}.$$

Finally, if f has a second derivative which is measurable and bounded on $(0, \infty)$, it can be seen from the Voronovskaja-type theorem for L_n due to Totik [20] and the dominated convergence theorem

$$\lim_{n \to \infty} n\{L_n^\alpha(f, x) - T^\alpha(f, x)\} = \int_0^\infty f''(\theta)(1+\theta)^2\, h_\alpha^x(\theta)\, d\theta,$$

whenever $x > 0$ and $0 < \alpha < \frac{1}{3}$.

In the next two sections we provide further approximation properties for both L_n^α and T^α. Section 2 is devoted to preservation of Lipschitz constants. In Section 3 we deal with the property of monotonic convergence under convexity. In both cases probabilistic methods play an important role.

2. Lipschitz Constants

Denote by $\mathrm{Lip}_I(A, \mu)$ the set of all real functions on the interval I such that

$$|f(x) - f(y)| \leqslant A\, |x-y|^\mu, \qquad x, y \in I,$$

where $A > 0$ and $\mu \in (0, 1]$.

THEOREM 1. *Let f be a real uniformly continuous function on $(0, \infty)$ such that $T^\alpha(|f|, x) < \infty$, for all $0 < \alpha < 1$ and $x > 0$. Then, the two following statements are equivalent:*

(a) $f \in \mathrm{Lip}_{(0,\infty)}(A, \mu)$.

(b) $T^\alpha f \in \mathrm{Lip}_{(0,\infty)}(A_{\alpha,\mu}, \mu)$, *for all* $\alpha \in (0, 1)$, *where*

$$A_{\alpha,\mu} := A(1-\alpha)^{-\mu}. \quad (11)$$

Proof. For $0 < x < y$ let (U, V) be a two-dimensional random vector having the density

$$h(\theta, \eta) := \frac{\Gamma((1+y)/\alpha)}{\Gamma(x/\alpha)\,\Gamma((y-x)/\alpha)\,\Gamma(1/\alpha)} \frac{\theta^{x/\alpha - 1}\eta^{(y-x)/\alpha - 1}}{(1+\theta+\eta)^{(1+y)/\alpha}}, \qquad \theta > 0,\ \eta > 0,$$

where $\Gamma(\cdot)$ is the gamma function. It is not hard to see that U (resp. $U+V$) has the inverse beta distribution with density h_α^x (resp. h_α^y) and hence, if $f \in \text{Lip}_{(0,\infty)}(A, \mu)$, we can write, using Jensen's inequality,

$$\begin{aligned} |T^\alpha(f, x) - T^\alpha(f, y)| &= |Ef(U) - Ef(U+V)| \\ &\leqslant E\,|f(U) - f(U+V)| \\ &\leqslant AE\,|V|^\mu \\ &\leqslant A(EV)^\mu \\ &= \frac{A}{(1-\alpha)^\mu}(y-x)^\mu, \end{aligned}$$

whenever $0 < \alpha < 1$. This shows (a) implies (b). The converse implication follows from (5), since $T^\alpha(f, x) \to f(x)$ (as $\alpha \to 0$) for all $x > 0$.

The proof above uses the "splitting" method due to Khan and Peters [12]. For an alternative proof, see Remark 3 at the end of the next section.

THEOREM 2. *Let f be a real uniformly continuous function on $[0, \infty)$. Then, the two following statements are equivalent:*

(a) $f \in \text{Lip}_{[0,\infty)}(A, \mu)$.

(b) $L_n^\alpha f \in \text{Lip}_{[0,\infty)}(A_{\alpha,\mu}, \mu)$, *for all* $\alpha \in (0, 1)$ *and* $n = 1, 2, \ldots$, *where* $A_{\alpha,\mu}$ *is defined in* (11).

Proof. If $f \in \text{Lip}_{[0,\infty)}(A, \mu)$ then $L_n f \in \text{Lip}_{[0,\infty)}(A, \mu)$, for all $n = 1, 2, \ldots$ (cf. [15]). Therefore, $L_n^\alpha f \in \text{Lip}_{[0,\infty)}(A_{\alpha,\mu}, \mu)$, for all $\alpha \in (0, 1)$ and $n = 1, 2, \ldots$, as follows from (7), Theorem 1, and the continuity of $L_n^\alpha f$. As for the converse implication, it is enough to observe that $L_n^\alpha(f, x) \to f(x)$ (as $\alpha \to 0$ and $n \to \infty$) for all $x \geqslant 0$, as a consequence of (5) and (9).

3. MONOTONIC CONVERGENCE

The main results in this section are the following:

THEOREM 3. *Let $x > 0$ and $1 > \alpha_1 > \alpha_2 > 0$. If f is a real convex function on $(0, \infty)$ such that $T^\alpha(|f|, x) < \infty$, for $\alpha = \alpha_1, \alpha_2$, then*

$$T^{\alpha_1}(f, x) \geqslant T^{\alpha_2}\left(f\left(\frac{1-\alpha_2}{1-\alpha_1}u\right), x\right).$$

If, in addition, f is nondecreasing, then

$$T^{\alpha_1}(f, x) \geqslant T^{\alpha_2}(f, x). \tag{12}$$

THEOREM 4. *Let $x > \alpha_1 > \alpha_2 > 0$. If f is a real function on $(0, \infty)$ such that $f(1/u)$ is convex and $T^{\alpha}(|f|, x) < \infty$, for $\alpha = \alpha_1, \alpha_2$, then*

$$T^{\alpha_1}(f, x) \geqslant T^{\alpha_2}\left(f\left(\frac{x-\alpha_1}{x-\alpha_2} u\right), x\right).$$

If, in addition, $f(1/u)$ is nondecreasing, then (12) *holds true.*

Remark 1. In general, (12) does not hold if the nondecreasing character of the convex function f is dropped. Take, for instance, $f(\theta) := -\theta$, $\theta > 0$. Then

$$T^{\alpha}(f, x) = \frac{x}{\alpha - 1}, \qquad \alpha \in (0, 1),\ x > 0.$$

A more interesting example is the following: Fix $x > 0$ and $1 > \alpha_1 > \alpha_2 > 0$. For each $t \geqslant 0$ let f_t be the function defined by

$$f_t(\theta) := e^{-t\theta}, \qquad \theta > 0.$$

We claim that

$$T^{\alpha_1}(f_r, x) > T^{\alpha_2}(f_r, x), \qquad \text{for some} \quad r > 0$$

and

$$T^{\alpha_1}(f_s, x) < T^{\alpha_2}(f_s, x), \qquad \text{for some} \quad s > 0.$$

In order to see this, observe that, for every $\alpha \in (0, 1)$, the function ϕ_{α}^{x} defined by

$$\phi_{\alpha}^{x}(t) := T^{\alpha}(f_t, x) = \int_0^{\infty} e^{-t\theta}\, dF_{\alpha}^{x}(\theta), \qquad t \geqslant 0,$$

is the Laplace transform of F_{α}^{x}, the inverse beta distribution function with parameters x/α, $1/\alpha$. By Fubini's theorem, we have

$$\int_0^{\infty} \phi_{\alpha}^{x}(t)\, e^{-t}\, dt = \int_0^{\infty} \frac{1}{1+\theta}\, dF_{\alpha}^{x}(\theta) = \frac{1}{1+x}, \qquad \alpha \in (0, 1).$$

Therefore, the assumption $\phi_{\alpha_1}^{x} \geqslant \phi_{\alpha_2}^{x}$, together with the continuity of these functions, implies $\phi_{\alpha_1}^{x} = \phi_{\alpha_2}^{x}$ and, from the uniqueness theorem for Laplace transforms, we have $F^x_{\alpha_1} = F^x_{\alpha_2}$, which is obviously false. Similarly, the assumption $\phi^x_{\alpha_1} \leqslant \phi^x_{\alpha_2}$ leads us to a contradiction. The claim is shown.

Notwithstanding, Theorem 4 above provides a wide class of nonincreasing convex functions for which (12) does hold. In fact, if $f(1/u)$ is convex and nondecreasing on $(0, \infty)$ then f is convex and nonincreasing. The converse is not true (take $f(u) := \log(1/u)$), but it is easy to see the following: If f is a nonincreasing function on $(0, \infty)$ which is twice differentiable and satisfies

$$2f'(u) + uf''(u) \geqslant 0, \qquad u > 0,$$

then $f(1/u)$ is a nondecreasing convex function. The remark is finished.

The proofs of Theorems 3 and 4 are based upon some properties concerning gamma processes. A stochastic process $(U_t)_{t \geqslant 0}$ starting at the origin, with stationary, independent increments, such that, for each $s > 0$, the density of U_s is given by

$$d_s(\theta) := \frac{1}{\Gamma(s)} \theta^{s-1} e^{-\theta}, \qquad \theta > 0,$$

is called a gamma process. It is clear that a gamma process is continuous in probability and, therefore, we assume, without loss of generality, that it has right-continuous paths [3, 19].

LEMMA 1. *Let $(U_t)_{t \geqslant 0}$ be a stochastic process starting at the origin, with nonnegative, integrable, stationary, independent increments, having right-continuous paths. For all $0 < r \leqslant s$ we have*

$$E(U_r \mid U_s) = \frac{r}{s} U_s, \qquad a.s.,$$

where $E(\cdot \mid \cdot)$ denotes conditional expectation.

Proof of Lemma 1. Let $s > 0$ and $n = 1, 2, \ldots$. We can write $U_s = \sum_{k=1}^{n} (U_{(k/n)s} - U_{((k-1)/n)s})$, as a sum of n integrable, independent, identically distributed random variables. Therefore (cf. [1])

$$E(U_{(k/n)s} \mid U_s) = \frac{k}{n} U_s, \qquad \text{a.s., } k = 1, \ldots, n. \tag{13}$$

If $0 < r < s$, consider a sequence of rational numbers $(q_n)_{n \geqslant 1}$ converging to r/s and such that $1 \geqslant q_n \geqslant r/s$ $(n \geqslant 1)$. By (13),

$$E(U_{q_n s} \mid U_s) = q_n U_s, \qquad \text{a.s., } n \geqslant 1. \tag{14}$$

Since $(U_t)_{t\geqslant 0}$ has right-continuous paths, the conclusion follows from (14) and the Lebesgue dominated convergence theorem for conditional expectations.

Let $(U_t)_{t\geqslant 0}$ and $(V_t)_{t\geqslant 0}$ be two independent gamma processes defined on the same probability space. For all $x>0$ and $t>0$, the random variable Y_t^x defined by

$$Y_t^x := \frac{U_{xt}}{V_t} \tag{15}$$

has the inverse beta distribution with parameters xt, t, as is easily checked. With this notation, we give the following

LEMMA 2. (a) *For all* $1<r\leqslant s$ *and* $x>0$, *we have*

$$E(Y_r^x \mid Y_s^x) = \frac{r}{s}\frac{s-1}{r-1}\, Y_s^x, \qquad a.s. \tag{16}$$

(b) *If* $1/x<r\leqslant s$, *then*

$$E\left(\frac{1}{Y_r^x}\,\middle|\, Y_s^x\right) = \frac{r}{s}\frac{xs-1}{xr-1}\frac{1}{Y_s^x}, \qquad a.s. \tag{17}$$

Proof of Lemma 2. Let $1<r\leqslant s$ and $x>0$. Since the random vectors (U_{xr}, U_{xs}) and (V_r, V_s) are independent we have

$$\begin{aligned} E(Y_r^x \mid U_{xs}, V_s) &= E(U_{xr} \mid U_{xs})\, E(V_r^{-1} \mid V_s) \\ &= \frac{r}{s} U_{xs} E(V_r^{-1} \mid V_s), \qquad \text{a.s.,} \end{aligned} \tag{18}$$

where the last equality follows from Lemma 1. On the other hand, the random variables $V_s V_r^{-1}$ and V_s are independent [11]. Therefore,

$$\begin{aligned} E(V_r^{-1} \mid V_s) &= V_s^{-1} E(V_s V_r^{-1} \mid V_s) \\ &= V_s^{-1} E(V_s V_r^{-1}) \\ &= V_s^{-1}(1+E((V_s-V_r)\, V_r^{-1})) \\ &= V_s^{-1}(1+E(V_s-V_r)\, EV_r^{-1}) \\ &= V_s^{-1}\left(1+\frac{s-r}{r-1}\right), \qquad \text{a.s.} \end{aligned} \tag{19}$$

From (18) and (19) we have

$$E(Y_r^x \mid U_{xs}, V_s) = \frac{r}{s} \frac{s-1}{r-1} Y_s^x, \qquad \text{a.s.} \tag{20}$$

Thus, statement (a) follows by taking in (20) the conditional expectation with respect to Y_s^x. Statement (b) is proved in a similar way.

Proof of Theorem 3. Comparing (2) and (15), it is clear that

$$T^\alpha(f, x) = Ef(Y_{\alpha^{-1}}^x). \tag{21}$$

Under the assumptions in Theorem 3, we have from (16), (21), and the conditional version of Jensen's inequality

$$\begin{aligned} T^{\alpha_1}(f, x) &= E(E(f(Y_{\alpha_1^{-1}}^x) \mid Y_{\alpha_2^{-1}}^x)) \\ &\geqslant Ef(E(Y_{\alpha_1^{-1}}^x \mid Y_{\alpha_2^{-1}}^x)) \\ &= Ef\left(\frac{1-\alpha_2}{1-\alpha_1} Y_{\alpha_2^{-1}}^x\right) \\ &= T^{\alpha_2}\left(f\left(\frac{1-\alpha_2}{1-\alpha_1} u\right), x\right). \end{aligned}$$

The conclusion follows.

Proof of Theorem 4. The proof runs along the lines of those of Theorem 3, using (17) instead of (16), and therefore we omit it.

Remark 2. The properties of the operators L_n^α concerning monotonic convergence can be summarized as follows: For $\alpha > 0$, $x \geqslant 0$, and $n = 1, 2, \ldots$,

$$L_n^\alpha(f, x) \geqslant L_{n+1}^\alpha(f, x),$$

whenever f is a nonincreasing convex function on $[0, \infty)$. (This was shown in the Introduction.) Moreover, in view of (7) and Theorems 3 and 4 above, we have for a fixed n,

$$L_n^{\alpha_1}(f, x) \geqslant L_n^{\alpha_2}(f, x),$$

whenever one of the following conditions is fulfilled:

(a) $1 > \alpha_1 > \alpha_2 > 0$ and $L_n f$ is convex and nondecreasing on $(0, \infty)$.

(b) $x > \alpha_1 > \alpha_2 > 0$ and $L_n(f, 1/\theta)$ is convex and nondecreasing on $(0, \infty)$.

Remark 3. Coming back to Theorem 1, an alternative proof for (a) implies (b) can be supplied by combining the representation given in (21) and Jensen's inequality. Actually, if $f \in \mathrm{Lip}_{(0,\infty)}(A, \mu)$, $0<\alpha<1$, and $0<x<y$, we have

$$\begin{aligned} |T^{\alpha}(f,x)-T^{\alpha}(f,y)| &= |Ef(Y^{x}_{\alpha-1})-Ef(Y^{y}_{\alpha-1})| \\ &\leqslant E\,|f(Y^{x}_{\alpha-1})-f(Y^{y}_{\alpha-1})| \\ &\leqslant AE\,|Y^{y-x}_{\alpha-1}|^{\mu} \\ &\leqslant A(EY^{y-x}_{\alpha-1})^{\mu} \\ &= \frac{A}{(1-\alpha)^{\mu}}(y-x)^{\mu}. \end{aligned}$$

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